

0. Matrix form မှာပေးထားတာ

$$A \begin{bmatrix} m \\ c \end{bmatrix} \approx b$$

↳ မသိသေးတာတွေကိုရှာဖွေရ

Normal equation

1. အသေးဆုံးသွယ်ကိန်းကို $A^T A$ နှင့် $A^T b$
2. Solve $A^T A x = A^T b$ (အကယ်၍)
 - ၁) backward sub
 - ၂) Cholesky factorization
3. ရှာဖွေရမယ့် x ($\begin{bmatrix} m \\ c \end{bmatrix}$) ကို အကဲခတ်ရမယ့် line

Given data points (x, y) : $(0, 1)$, $(1, -1)$, $(2, 4)$, $(3, 2)$. We want to fit these 4 points with the line $y = mx + c$. Find the line that best fits the data points in least-squares sense using the method of normal equations.

We want m and c satisfying:

$$\begin{aligned} mx_1 + c &\approx y_1 \\ mx_2 + c &\approx y_2 \\ mx_3 + c &\approx y_3 \\ mx_4 + c &\approx y_4 \end{aligned}$$

$$\begin{aligned} m(0) + c &\approx 1 \\ m(1) + c &\approx -1 \\ m(2) + c &\approx 4 \\ m(3) + c &\approx 2 \end{aligned}$$

Matrix form:

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \\ x_4 & 1 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} \approx \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} \approx \begin{bmatrix} 1 \\ -1 \\ 4 \\ 2 \end{bmatrix}$$

$$A \begin{bmatrix} m \\ c \end{bmatrix} \approx b$$

By the method of normal equations:

1. အသေးဆုံးသွယ်ကိန်းကို $A^T A$ နှင့် $A^T b$

$$\begin{aligned} \textcircled{1} A^T A &= \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0^2 + 1^2 + 2^2 + 3^2 & 0 + 1 + 2 + 3 \\ 0 + 1 + 2 + 3 & 1 + 1 + 1 + 1 \end{bmatrix} \\ &= \begin{bmatrix} 14 & 6 \\ 6 & 4 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \textcircled{2} A^T b &= \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 4 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 0(1) + 1(-1) + 2(4) + 3(2) \\ 1(1) + 1(-1) + 1(4) + 1(2) \end{bmatrix} \\ &= \begin{bmatrix} 13 \\ 6 \end{bmatrix} \end{aligned}$$

2. Solve $A^T A x = A^T b$ using Cholesky factorization

$$\begin{bmatrix} 14 & 6 \\ 6 & 4 \end{bmatrix} x = \begin{bmatrix} 13 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 14 & 6 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} 13 \\ 6 \end{bmatrix}$$



Cholesky Factorization $\begin{bmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} \\ 0 & l_{22} \end{bmatrix} = \begin{bmatrix} l_{11}^2 & l_{11}l_{21} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 \end{bmatrix}$

$A = LL^T$

$l_{11}^2 = a_{11} \Rightarrow l_{11} = \pm \sqrt{a_{11}}$ By convention, we always choose the **positive** values for the diagonal entries of L .

$l_{11}l_{21} = a_{21} \Rightarrow l_{21} = a_{21}/l_{11}$ $l_{21}^2 + l_{22}^2 = a_{22} \Rightarrow l_{22} = \pm \sqrt{a_{22} - l_{21}^2}$

Example

Compute the Cholesky factorization of:

$$\begin{bmatrix} 9 & -15 \\ -15 & 26 \end{bmatrix}$$

$$l_{11} = \sqrt{a_{11}} = \sqrt{9} = 3$$

$$l_{21} = a_{21}/l_{11} = -15/3 = -5$$

$$l_{22} = \sqrt{a_{22} - l_{21}^2} = \sqrt{26 - (-5)^2} = \sqrt{26 - 25} = 1 \therefore L = \begin{bmatrix} 3 & 0 \\ -5 & 1 \end{bmatrix}$$

Done

Solving Linear System with Cholesky

To solve $Ax = b$ when A is symmetric positive definite:

1. Factor $A = LL^T$
2. Solve $Lw = b$ for w (forward substitution)
3. Solve $L^T x = w$ for x (back substitution)

Lecture 10: Linear Least Square

အဲဒါ solve linear least square (အကဲခတ်ရမယ့် best-fit ကိန်း)

$$\begin{bmatrix} 14 & 6 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} 13 \\ 6 \end{bmatrix}$$

→ $A^T A$: symmetric positive definite,
write $A^T A = LL^T$

$$l_{11} = \sqrt{14}$$

$$l_{21} = \frac{6}{l_{11}} = \frac{6}{\sqrt{14}}$$

$$l_{22} = \sqrt{4 - \frac{6^2}{14}} = \sqrt{4 - \left(\frac{6}{\sqrt{14}}\right)^2} = \sqrt{\frac{10}{7}}$$

$$\left\{ \begin{bmatrix} \sqrt{14} & 0 \\ \frac{6}{\sqrt{14}} & \sqrt{\frac{10}{7}} \end{bmatrix} \right.$$

Solving Linear System with Cholesky

To solve $Ax = b$ when A is symmetric positive definite:

~~1. Factor $A = LL^T$~~

2. Solve $Lw = b$ for w (forward substitution)

3. Solve $L^T x = w$ for x (back substitution)

2.2 Solve $Lw = b$ for w

$$\begin{bmatrix} \sqrt{14} & 0 \\ \frac{6}{\sqrt{14}} & \sqrt{\frac{10}{7}} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 13 \\ 6 \end{bmatrix}$$

$$\therefore w_1 = \frac{13}{\sqrt{14}}$$

$$\frac{6}{\sqrt{14}} \left(\frac{13}{\sqrt{14}} \right) + \sqrt{\frac{10}{7}} w_2 = 6$$

$$\frac{78}{14} + \sqrt{\frac{10}{7}} w_2 = 6$$

$$w_2 = 6 - \frac{78}{14} \div \sqrt{\frac{10}{7}}$$

$$\therefore w_2 \approx 0.358569$$

2.3 Solve $L^T x = w$ for x

$$\begin{bmatrix} \sqrt{14} & \frac{6}{\sqrt{14}} \\ 0 & \sqrt{\frac{10}{7}} \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} \frac{13}{\sqrt{14}} \\ 0.358569 \end{bmatrix}$$

$$c = 0.3$$

$$m = 0.8$$

$$\therefore y = 0.8x + 0.3 \quad \#$$